A Fast Solver for the Stokes Equations

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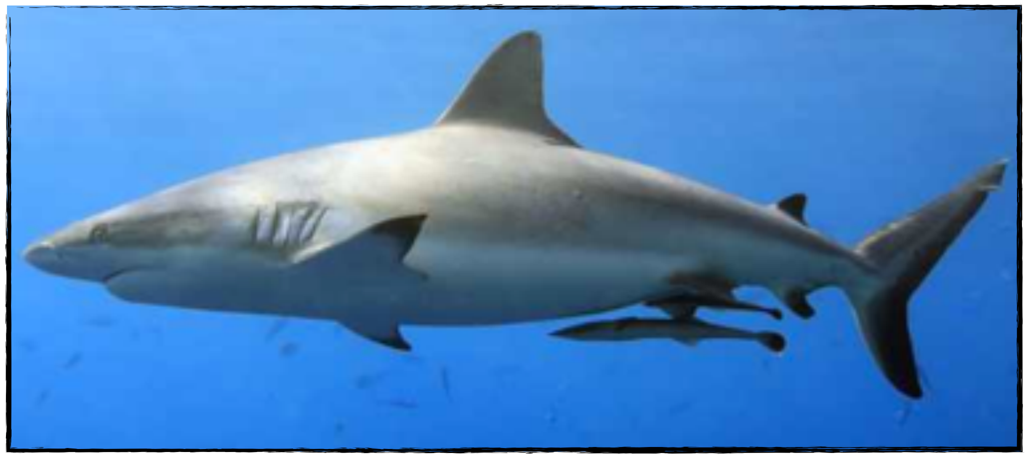
CS 598APK, Fall 2017

Stokes Equations?



http://www.earthtimes.org/

wallpapers-xs.blogspot.com

http://testingstufftonight.blogspot.com/

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0$$

Navier Stokes

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial[\rho u_i u_j]}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \rho f_i$$

Input

ρ

Resistance μ

→

Initial Conditions (t)

Boundary Conditions (**x**)

Solution Algorithm: **x**, t

→

Output

$\vec{u}(\vec{x}, t)$

$p(\vec{x}, t)$

Stokes Equations?

https://www.promegaconnections.com/

$$\mu \to \infty$$

Continuity $$\frac{\partial(\rho u_i)}{\partial x_i} = 0$$

Stokes PDE : Especially useful in small scales

$$0 = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + f_i$$

https://www.youtube.com/watch?v=p08_KlTKP50

UNM Physics and Astronomy, Sped up 10x

No time dependence - completely reversible

Linear PDEs :) $\longrightarrow$ Integral Equations

u : A system of linear PDEs :(

P : A coupled system of PDES :’((

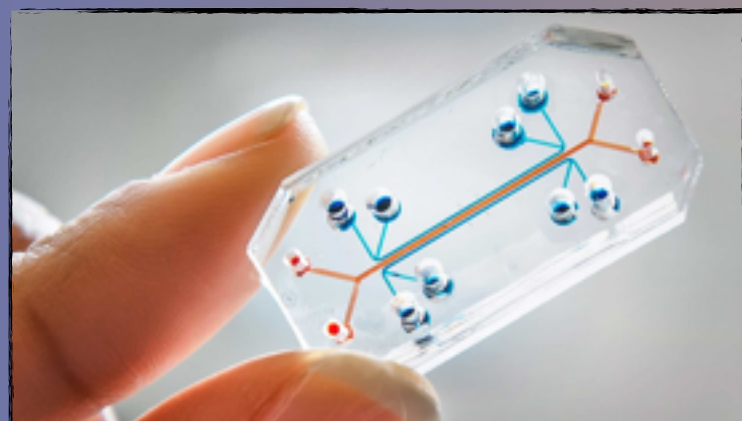
ρ μ → Boundary Conditions (x) Solution Algorithm → $\vec{u}(\vec{x})$ $P(\vec{x})$

Necessity ?

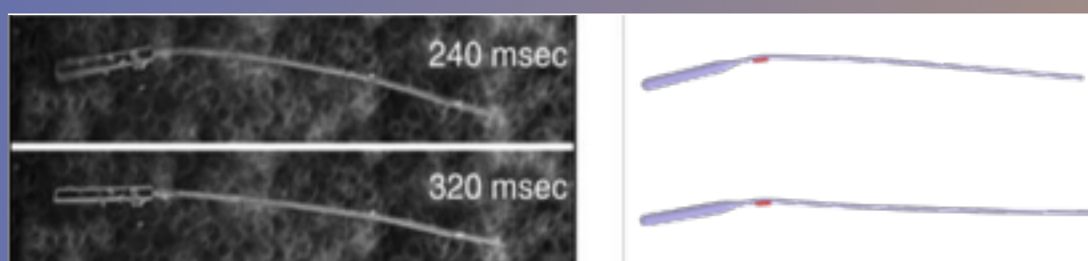
Time

Stokes Flow

Present



http://www.elveflow.com/



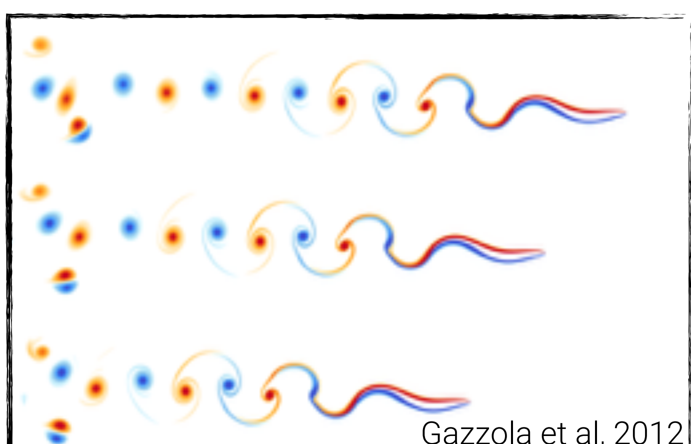
Xiaotian et al, Preprint

Goal

http://techgenmag.com/

Viscous flow

Gazzola et al, 2012



Gazzola et al, 2012

+ Integral Equations & Fast Algorithms

Why Integral Equation techniques?

$$\mu\nabla^2\mathbf{u} - \boldsymbol{\nabla} P + \mathbf{f} = \mathbf{0}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

Challenges	FDM	FEM	BEM
4 unknowns @ **x**	X	X	X
Continuity	X Needs projection	X Needs projection	√ Identically satisfied
Discretisation spaces	X What spaces?	X inf-sup restriction	√ Any meaningful rep.
Conditioning	X Very Bad	X Bad: Preconditioners	√ Good κ indep. of problem size
Time to Solution	X Slow	X Slow	√ Fast O(n)
Higher order	X Difficult	X Difficult	√ Not difficult

C Pozrikidis, 1992   Malhotra et al, 2014   Klinteberg et al, 2016

Procedure

Construct representation : Integral Operators + Potential Theory
BVP & IE solution existence/uniqueness
IE discretization
Quadrature Rule
Time progression

Constructing a representation - some theory

Wikipedia

Erik Ivar Fredholm

$$\mu\nabla^2\mathbf{u} - \boldsymbol{\nabla}P + \mathbf{g}\delta(\mathbf{x}-\mathbf{x_o}) = \mathbf{0}$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\mathcal{D}}[\Gamma, \mathbf{q}](\mathbf{x})$$

Double Layer Potential Source curve Hydrodynamic potential

Fluid stress

$$\sigma_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Laplace PDE

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r}$$

Stokes PDE

$$u_i = G_{ij}g_j \quad P = \mu p_j g_j \quad \sigma_{ij} = \mu T_{ijk}g_j$$

$$G_{ij}(\mathbf{x},\mathbf{y}) = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \quad p_j(\mathbf{x},\mathbf{y}) = 2\frac{\tilde{x}_j}{r^3} \quad T_{ijk}(\mathbf{x},\mathbf{y}) = -6\frac{\tilde{x}_i\tilde{x}_j\tilde{x}_k}{r^5}$$

Stokeslet Stresslet

$$K_D(\mathbf{x},\mathbf{y}) = \hat{n}.\nabla_y G(\mathbf{x},\mathbf{y})$$

$$K_j^D(\mathbf{x},\mathbf{y}) = T_{ijk}(\mathbf{x},\mathbf{y})n_k(\mathbf{y})$$

Stresslet

Kernels to construct solution exist

C Pozrikidis, 1992

Constructing a representation - some theory

$$\mu \nabla^2 \mathbf{u} - \boldsymbol{\nabla} P + \mathbf{g}\delta(\mathbf{x} - \mathbf{x_o}) = \mathbf{0}$$

Laplace PDE

Multipole

$$G(\mathbf{x}, \mathbf{y}) \Rightarrow \hat{a} \cdot \frac{\partial G}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y})$$

$$\int_U (\psi\, \Delta\varphi - \varphi\, \Delta\psi)\; dV = \oint_{\partial U} \left(\psi \frac{\partial \varphi}{\partial \mathbf{n}} - \varphi \frac{\partial \psi}{\partial \mathbf{n}} \right) dS \longrightarrow (\mathcal{S}(\hat{n} \cdot \nabla u) - \mathcal{D}u)(x) = u(x)$$

Stokes PDE

For **u** : Stokeslet, Stokeslet Doublet, Stokeslet Quadrupole

$$\frac{\partial}{\partial x_j}(u_i' \sigma_{ij} - u_i \sigma_{ij}') = u_i' \frac{\partial \sigma_{ij}}{\partial x_j} - u_i \frac{\partial \sigma_{ij}'}{\partial x_j} \longrightarrow u_j(\mathbf{x}) = -\frac{1}{8\pi\mu} \int_\Gamma G_{ij}(\mathbf{y}, \mathbf{x}) f_i(\mathbf{y}) dS(\mathbf{y}) + \frac{1}{8\pi} \int_\Gamma T_{ijk}(\mathbf{y}, \mathbf{x}) n_k(\mathbf{y}) u_i(\mathbf{y}) dS(\mathbf{y})$$

Reciprocal Identity : Strong physical meaning

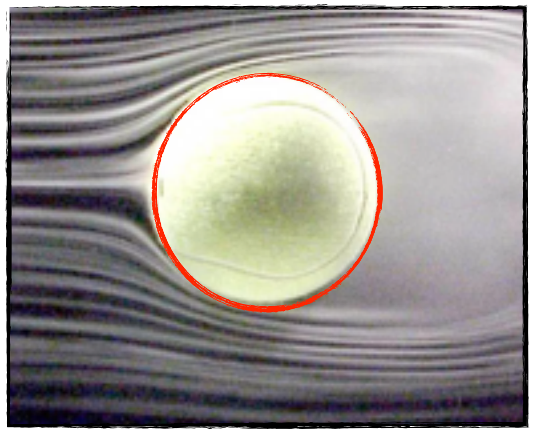
The Stokeslet and Stresslet provide a complete representation

C Pozrikidis 2.3.10, 1992

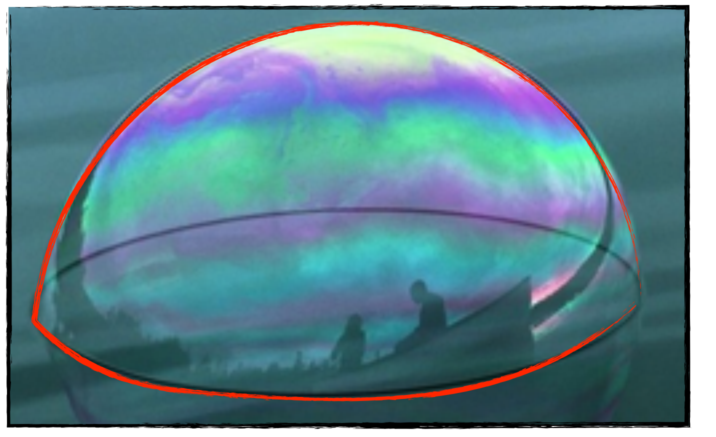
Boundary Value Problems

Typically interested in external problems : **Dirichlet** and Neumann

Prescribed velocity



www.livescience.com



www.exposureguide.com

Prescribed force

Laplace PDE Null space for external Dirichlet : Fredholm Alternative from int. Neumann

Stokes PDE Null space for external Dirichlet : Fredholm Alternative from int. Neumann :(

C Pozrikidis, Thm 4.7.1 1992

$$u_i(\mathbf{x}) = \int_\Gamma T_{jik}(\mathbf{y},\mathbf{x}) n_k(\mathbf{y}) q_j(\mathbf{y}) dS(\mathbf{y}) + \mathcal{V}_i(\mathbf{x})$$

$\mathcal{V}_i(\mathbf{x})$ Compensate for deficiency in range Usually Prescribed (or) Single Layer op.

$\therefore$ At surface,

$$u_i^d(\mathbf{x}) + V_i + \epsilon_{ijk}\Omega_j X_{0,k} = 4\pi q_i(\mathbf{x}) + PV \int_\Gamma T_{jik}(\mathbf{y},\mathbf{x}) n_k(\mathbf{y}) q_j(\mathbf{y}) dS(\mathbf{y}) + \mathcal{V}_i(\mathbf{x})$$

Deformation Translation Rotation

Still **(I+ Compact)** : Well conditioned

BVP straightforward?

C Pozrikidis, 1992

BVPs : Mobility and Resistance

$\mathcal{V}_i(\mathbf{x})$ complicates the problem, leading to a dichotomy:

$$u_i^d(\mathbf{x}) + V_i + \epsilon_{ijk}\Omega_j X_{0,k} = 4\pi q_i(\mathbf{x}) + PV \int_\Gamma T_{jik}(\mathbf{y},\mathbf{x}) n_k(\mathbf{y}) q_j(\mathbf{y}) dS(\mathbf{y}) + \mathcal{V}_i(\mathbf{x})$$

Deformation Translation Rotation

Still **(I+ Compact)** : Well conditioned

Resistance problem

$$(\mathbf{V}, \mathbf{\Omega}) \overset{\text{Linear}}{\Rightarrow} (\mathbf{f}, \mathbf{t})$$

If prescribed motion, find forces

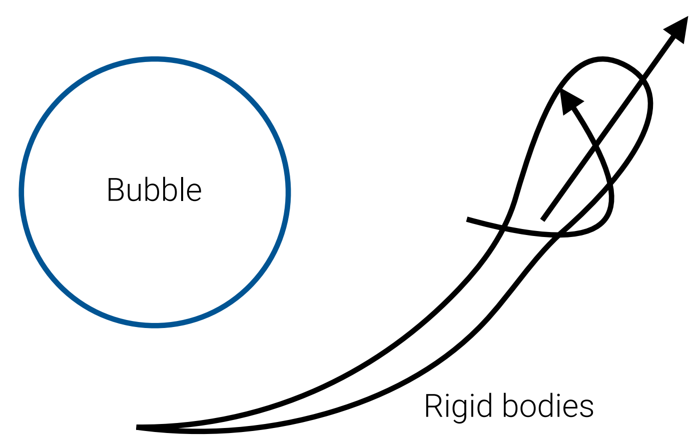
Mobility problem

$$(\mathbf{f}, \mathbf{t}) \overset{\text{Linear}}{\Rightarrow} (\mathbf{V}, \mathbf{\Omega})$$

If prescribed forces, find motion

This leads to (additional) constraints in some cases:

Gazzola et al, 2013



C Pozrikidis, 1992

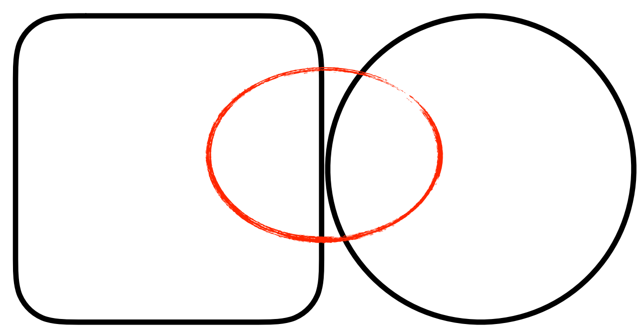
Discretization & Solution

1. Surface and function discretisation requirements as required

2. QBX to calculate matrix coefficients of discrete system to be solved (accelerated by precomputing/FMM)

IE Discretisation   **Nystrom carries over: Approximate quadrature sufficient for off surface evaluation**

Now use QBX (with trapz/gauss) to calculate PV of DLP on the surface



Nearly singular evaluations: Expansion may fail

3. Enforce BC at quadrature points to solve linear system A**q** = b by GMRES (const iter.)

4. With **q** obtained, get **u** on domain using DLP (FMM accelerated)

5. Calculate *p* or ***σ*** as a post processing step, as needed

6. Get new particle positions using force history and some time stepping scheme

C Pozrikidis, 1992

Conclusions

We have an IE method to solve the Stokes flow problem

Similarities/ Differences to Laplace PDE

Optimal (or) near optimal time

Numerical experiments to be conducted

Any questions?

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3. Malhotra, Dhairya, Amir Gholami, and George Biros. "A volume integral equation stokes solver for problems with variable coefficients." Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis. IEEE Press, 2014.
4. af Klinteberg, Ludvig, and Anna-Karin Tornberg. "A fast integral equation method for solid particles in viscous flow using quadrature by expansion."
5. Journal of Computational Physics 326 (2016): 420-445.
6. Klöckner, Andreas, et al. "Quadrature by expansion: A new method for the evaluation of layer potentials." Journal of Computational Physics 252 (2013): 332-349.