Boundary Integrals and Shape Optimization: A match made in some nerdy mathematician heaven

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Why shape optimization?

- Optimization of structures is a popular "new" topic in mechanics
- Most work uses SIMP (Solid Isotropic Material with Penalization) model
- But this method does a bad job of defining material boundaries (bad for manufacturability)
- In shape optimization the boundaries are clearly defined (though it is harder to add voids to the interior)

SIMP example

Figure: Result of Shape Optimization using SIMP method

Why boundary integrals?

- With shape optimization, no need for interior discretization
- Only discretizing boundary leads to much smaller meshes and faster numerical solutions
- Can model complex exterior geometry for only a marginally increased cost
- Many optimization functions only need boundary data (or can be converted to boundary integrals)

How do we know if a structure is optimal?

Of course, there would be many ways to define what structure is the best. However a few metrics are used predominantly, depending on the type of problem to be analyzed.

- Compliance (Most common objective)
- Output displacement or force
- Maximum Stress
- Structural Stability
- Size of Bandgap
- Thermal Dissipation

The metrics define performance of a structure, but still need a way to evaluate that performance as the structure evolves. Thus we must turn to continuum mechanics to determine how a structure responds to given boundary conditions.

Governing differential equation

At a higher level, topology optimization can really just be viewed as an extension of continuum mechanics where some term in the governing differential equation that was previously fixed is now treated as a design variable. For example, in the case of 2-D statics, we have the strong form of the pde:

$$\begin{aligned} \mathrm{div}\sigma + b &= 0 && \text{in} \quad \Omega \\ u &= u^p && \text{on} \quad \Gamma^u \\ \sigma \cdot n &= t^t && \text{on} \quad \Gamma^t \end{aligned}$$

Don't want to solve the strong form, so we solve the boundary integral problem instead (assuming no body forces):

$$c_{ij}(\mathbf{x})u_j(\mathbf{x}) + \int_{\Gamma} T^*_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})d\Gamma(\mathbf{y}) = \int_{\Gamma} u^*_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})d\Gamma(\mathbf{y})$$

Boundary integral for 2D elasticity

$$c_{ij}(\mathbf{x})u_j(\mathbf{x}) + \int_\Gamma T^*_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})d\Gamma(\mathbf{y}) = \int_\Gamma u^*_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})d\Gamma(\mathbf{y})$$

- $c_{ij} = \frac{1}{2}\delta_{ij}$ for smooth problems, best not to try for nonsmooth problems ($= \lim_{r\to 0}\int_\Gamma T^*_{ij}d\Gamma$)
- T^*_{ij} is the fundamental traction kernel
- u^*_{ij} is the fundamental displacement kernel
- Dirichlet problems produce equation of the first kind
- Neumann problems produce equation of the second kind
- Displacements at any point in the domain can be obtained through Somigliana's identity

$$u_j(\mathbf{x}) + \int_\Gamma T^*_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})d\Gamma(\mathbf{y}) = \int_\Gamma u^*_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})d\Gamma(\mathbf{y})$$

How to optimize

- In the governing pde, we need to find a u that solves the problem.
- In topology optimization, we change some term in the pde in order to adjust u.
- Changing b (when included), u^p, or t^f is really just changing the definition of the problem.
- Often try to change $\mathbb{C}$ (SIMP method), but hard to do with boundary integrals
- Instead change Ω and Γ

Sensitivity formulation

- For efficient optimization, need to use gradient-based optimizer
- Sensitivity of any function f with respect to design variable α follows the form:

$$\frac{df}{d\alpha} = \frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \mathbf{u}}\frac{d\mathbf{u}}{d\alpha} + \frac{\partial f}{\partial \mathbf{t}}\frac{d\mathbf{t}}{d\alpha}$$

The partial derivative terms are easy to calculate. The sensitivies of **u** and **t** are trickier, need to solve an adjoint problem.

Adjoint problem

Original system looks like:

$$\mathbf{Hu} = \mathbf{Gt}$$

Differentiate with respect to α:

$$\mathbf{H}'\mathbf{u} + \mathbf{H}\mathbf{u}' = \mathbf{G}'\mathbf{t} + \mathbf{G}\mathbf{t}'$$

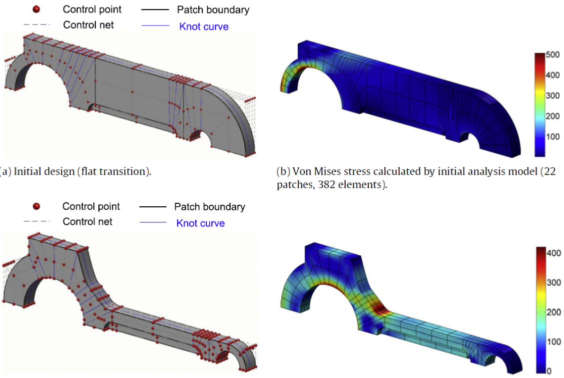
$\mathbf{H}'$ and $\mathbf{G}'$ can be calculated analytically, and $\mathbf{u}$ and $\mathbf{t}$ are known from solving BVP. Now we have a system where $\mathbf{u}$' and $\mathbf{t}$' are the unknowns, and operators are the same as in the original BVP (plus some constants).

Pulling it together

Process to optimize structure proceeds like

- Use boundary integral to solve BVP
- Reuse boundary integral to compute sensitivities of BVP wrt design variables (shape of domain)
- Feed sensitivities to an optimizer to update design at each iteration (Method of Moving Asymptotes is the usual choice in structural optimization)
- Repeat until "optimal"

Example



(a) Initial design (flat transition).

(b) Von Mises stress calculated by initial analysis model (22 patches, 382 elements).

(c) Optimized design (curved transition).

(d) Optimized analysis model and stress.

Figure: Result of Shape Optimization with Boundary Integral Solver