A direct solver for Poisson's equation using spectral element discretization

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November 29, 2017

Motivation & Background

- Combine the following two parts to get a direct solver
  - High-order spectral approximation methods
  - Hierarchical solver
- [mar, 2013]

Methodology

Within the paper, Martinsson

- used spectral collocation method
- presented performance and accuracy data

Plan for presentation

In this talk I will cover the following:

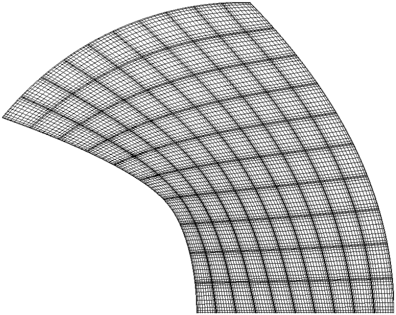
- Algorithm
- Numerical results

Introduction to spectral element method(SEM)

- Spectral element method(SEM) is in effect a high-order finite element method
- Basis functions: Lagrange polynomials on Gauss-Lobatto-Legendre(GLL) points
- Galerkin $\Longrightarrow$ test function space is the same as basis function

SEM triangulation

- Discretize the domain into a union of quadrilateral (squares or rectangles for simple cases)
- Example mesh



SEM formulation for Poisson's equation

- Poisson's equation with inhomogeneous Dirichlet boundary condition and no forcing term reads

$$\nabla^2 u = 0,\ u|_{\partial\Omega} = f$$

- Weak form

$$\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}V = 0$$

- or in matrix operator form

$$A\mathbf{u} = 0$$

Methodology: one element case

- Solving for the interior points:

$$\begin{aligned} A_{i,i}\mathbf{u}_i &= -A_{i,e}\mathbf{u}_e \\ \mathbf{u}_i &= -(A_{i,i})^{-1}A_{i,e}\mathbf{u}_e = U\mathbf{u}_e \end{aligned} \quad (1)$$

- Using derivative operators $D = \partial_x$ and $E = \partial_y$ to define Dirichlet-to-Neumann(DtN) operators that find the partial derivatives on the exterior points

$$\begin{aligned} \mathbf{v}_e &= (D_{e,e} + D_{e,i}U)\mathbf{u}_e = V\mathbf{u}_e \approx \partial_x u_e \\ \mathbf{w}_e &= (E_{e,e} + E_{e,i}U)\mathbf{u}_e = W\mathbf{u}_e \approx \partial_y u_e \end{aligned} \quad (2)$$

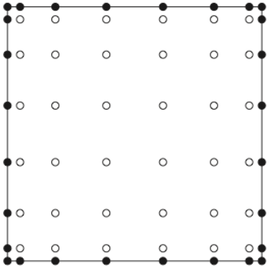


Figure: One element GLL points. Source: [mar, 2013]

Methodology: multiple element case

- Index sets when two spectral elements are side-by-side

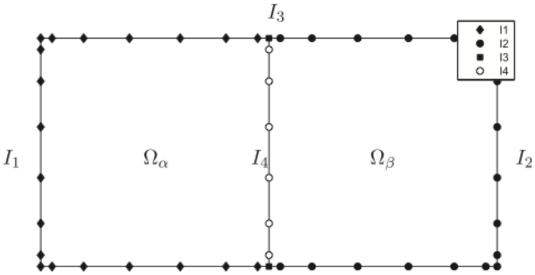


Figure: Index sets. Source: [mar, 2013]

Methodology: multiple element case

- Merging operation: from points on the boundaries of box α, β, find operators U, V, W for the combined box(ext. to int., ext. to ext.)
- I_4: interior points; I_1, I_2, I_3: exterior points
- Ordering in the following way

$$u_4 = U\mathbf{u}_e = U \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = V \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Methodology: multiple element case

- Box boundary equilibrium: normal derivatives match
- If aligned horizontally

$$U = (V^{\alpha}_{4,4} - V^{\beta}_{4,4})^{-1} \left[-V^{\alpha}_{4,1} | V^{\beta}_{4,2} | V^{\beta}_{4,3} - V^{\alpha}_{4,3} \right] \quad (3)$$

- If aligned vertically

$$U = (W^{\alpha}_{4,4} - W^{\beta}_{4,4})^{-1} \left[-W^{\alpha}_{4,1} | W^{\beta}_{4,2} | W^{\beta}_{4,3} - W^{\alpha}_{4,3} \right] \quad (4)$$

Methodology: multiple element case

- Next, find DtN operators V, W

$$V = \begin{bmatrix} V^{\alpha}_{1,1} & \mathbf{0} & V^{\alpha}_{1,3} \\ \mathbf{0} & V^{\beta}_{2,2} & V^{\beta}_{2,3} \\ \frac{1}{2}V^{\alpha}_{3,1} & \frac{1}{2}V^{\beta}_{3,2} & \frac{1}{2}V^{\alpha}_{3,3} + \frac{1}{2}V^{\beta}_{3,3} \end{bmatrix} + \begin{bmatrix} V^{\alpha}_{1,4} \\ V^{\beta}_{2,4} \\ \frac{1}{2}V^{\alpha}_{3,4} + \frac{1}{2}V^{\beta}_{3,4} \end{bmatrix} U \tag{5}$$

$$W = \begin{bmatrix} W^{\alpha}_{1,1} & \mathbf{0} & W^{\alpha}_{1,3} \\ \mathbf{0} & W^{\beta}_{2,2} & W^{\beta}_{2,3} \\ \frac{1}{2}W^{\alpha}_{3,1} & \frac{1}{2}W^{\beta}_{3,2} & \frac{1}{2}W^{\alpha}_{3,3} + \frac{1}{2}W^{\beta}_{3,3} \end{bmatrix} + \begin{bmatrix} W^{\alpha}_{1,4} \\ W^{\beta}_{2,4} \\ \frac{1}{2}W^{\alpha}_{3,4} + \frac{1}{2}W^{\beta}_{3,4} \end{bmatrix} U \tag{6}$$

Methodology: hierarchical scheme

Consider a square domain, construct a binary tree

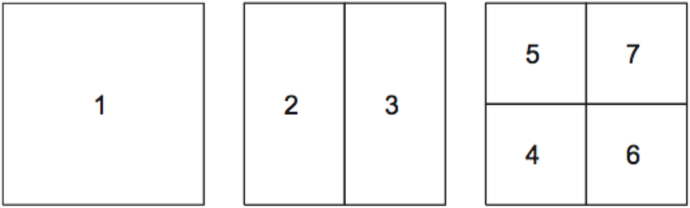


Figure: Box ids. Source: [mar, 2013]

Methodology: hierarchical scheme

Algorithm 1 Pre-computation(build)

for $\tau = N_{boxes}$ to 1 **do**
    if τ is a leaf **then**
        Eval $U^{\tau}, V^{\tau}, W^{\tau}$, Eqs: 1,2
    else
        Let σ_1, σ_2 be the children of τ
        if σ_1 and σ_2 are horizontal **then**
            Eval U^{τ} using $V^{\alpha,\beta}$, Eq 3
        else
            Eval U^{τ} using $W^{\alpha,\beta}$, Eq 4
        end if
        Eval V^{τ}, W^{τ}, Eqs: 5,6
    end if
end for

Methodology: hierarchical scheme

Algorithm 2 Forward solve

1: Find boundary data for box 1 $\mathbf{u} = f(\mathbf{x})$
2: **for** $\tau = 1$ to N_{boxes} **do**
3: $\quad \mathbf{u}(I_i^\tau) = U^\tau \mathbf{u}(I_e^\tau)$
4: **end for**

Verification case: problem setup

- On domain $[0, 1]^2$, function $u = \cos kx \exp ky$ is an exact solution to the Poisson's equation, and has nontrivial boundary values
- Take $k = \pi/2$

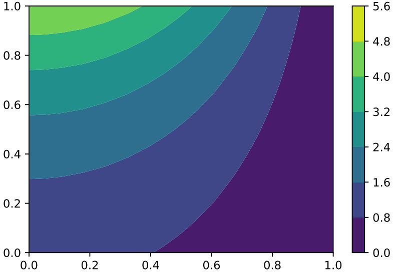


Figure: Exact solution

Verification case: series of solution

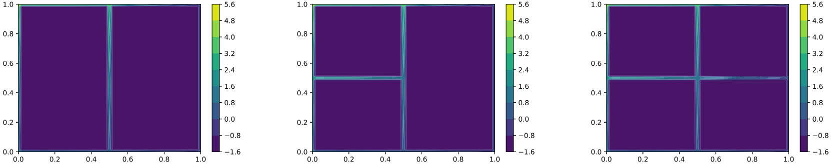


Figure: Procedural solution

Verification case: series of solution

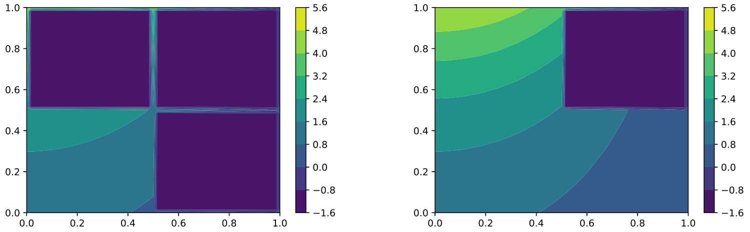


Figure: Procedural solution(continued)

Verification case: series of solution

The final solution and the error

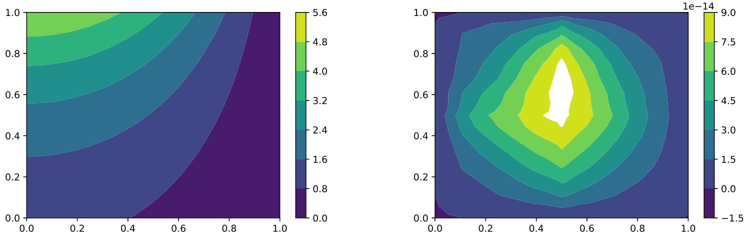


Figure: Final solution and error

Verification case: convergence

L2 norm error as N increases:

N	Direct solve	SEM inverse	D.o.f.
N=4	7.241619e-05	4.607383e-06	81
N=6	9.484310e-07	2.703998e-09	169
N=8	1.976962e-10	1.138766e-12	289
N=10	8.856289e-13	3.308540e-13	441
N=12	1.160633e-13	1.927155e-13	625

Timing: solve

Solution time wise, this algorithm is fairly competitive to solving normal SEM with CG

N	Direct(err)	SEM-Iter.(err)	Direct(time,s)	SEM-Iter.(time,s)
N=6	9.4843e-07	1.3211e-07	1.4010e-03	3.4709e-03
N=8	1.9770e-10	4.4944e-10	1.0770e-03	5.1010e-03
N=10	8.8563e-13	4.9215e-12	1.6313e-03	1.9790e-02
N=12	1.1606e-13	8.8005e-12	2.3076e-03	3.7499e-02

Conclusion

- Implemented a Poisson's equation solver using algorithm described

References I

A direct solver for variable coefficient elliptic pdes discretized via a composite spectral collocation method. *Journal of Computational Physics*, 242:460 – 479, 2013. ISSN 0021-9991.