Direct solution technique for frequency-domain scattering problems[1]

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CS598 Course Project, Fall 2017

[1] Gillman, A., Alex H. B., and Martinsson P.G. "A spectrally accurate direct solution technique for frequency-domain scattering problems with variable media." BIT Numerical Mathematics 55.1 (2015): 141-170

Introduction

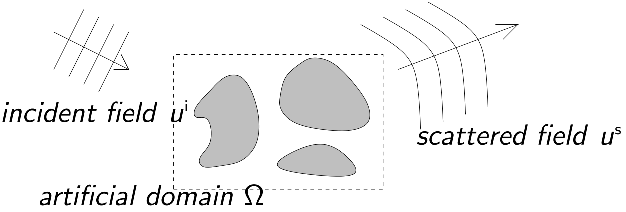


Figure: Schematic of the problem, $u = u^{\mathrm{s}} + u^{\mathrm{i}}$

- Compute the scattered wave u^{s}, given incident wave u^{i}
- Mathematically, the scattered field u^{s} satisfies the variable coefficient Helmholtz equation

$$\Delta u^{\mathrm{s}}(\boldsymbol{x}) + \kappa^2(1 - b(\boldsymbol{x}))u^{\mathrm{s}}(\boldsymbol{x}) = \kappa^2 b(\boldsymbol{x})u^{\mathrm{i}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2, \quad (1)$$

Sommerfeld radiation condition

$$\frac{\partial u^{\mathrm{s}}}{\partial r} - i\kappa u^{\mathrm{s}} = o(r^{-1/2}), \qquad r := |\boldsymbol{x}| \to \infty, \quad (2)$$

Motivation

- Time-harmonic wave equations are relevant for practical applications: photonics, acoustics, placing your WiFi router![2]

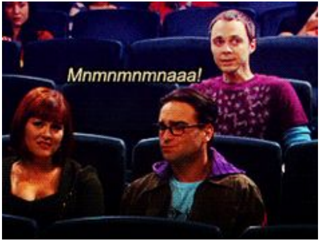


Figure: Adventures in the acoustics of movie theaters[3]

- Solution method proposed is spectrally accurate, robust and computationally efficient.

[2] http://www.caam.rice.edu/ gillmana/Wi-Fly.html

[3] https://i.pinimg.com/236x/9f/18/50/9f1850ce9a989ed19b1f9f86afebaacd–acoustic-asparagus.jpg

Overview

- Split problems into the interior Ω, and the exterior $\mathbb{R}^2 \setminus \Omega$ problems
- Prepare solution operators for both, 'glue' them at $\partial\Omega$ to get solution
- Use a tree of boxes to solve the interior variable coefficient problem (hierarchical merges)
- Interior Dirichlet problem with variable coefficient $b(\boldsymbol{x})$
  - DtN map: $T_{\text{int}} : T_{\text{int}} u = u_n \ \forall x \ \in \ \partial\Omega$
  - Issue: Discrete domain difference operator $\rightarrow$ norm scales as N!
- Exterior Dirichlet problem with Sommerfeld condition:
  - DtN map: $T_{\text{ext}} : T_{\text{ext}} u^s = u_n^s \ \forall x \ \in \ \partial\Omega$
  - Issue: Same as T_{int}
- Combine the two: $(T_{\text{int}} - T_{\text{ext}}) u^{\text{s}}|_{\partial\Omega} = u_n^{\text{i}} - T_{\text{int}} u^{\text{i}}|_{\partial\Omega}$
- Order of $(T_{\text{int}} - T_{\text{ext}})$ still $+1 \rightarrow$ Ill-conditioned system $\rightarrow$ use Impedance-to-impedance maps!

Impedance-to-Impedance maps

- We solve the interior variable coefficient problem:

$$\begin{aligned} [\Delta + \kappa^2(1 - b(\boldsymbol{x}))]u(\boldsymbol{x}) &= 0 \qquad \boldsymbol{x} \in \Omega \ , \qquad (3) \\ u_n + i\eta u|_{\partial\Omega} &= f \qquad \text{on } \partial\Omega \ , \qquad (4) \end{aligned}$$

- Use incoming and outgoing impedance boundary conditions (different from mixed boundary conditions!):

$$\begin{aligned} f &:= u_n + i\eta u|_{\partial\Omega} \qquad (5) \\ g &:= u_n - i\eta u|_{\partial\Omega} \qquad (6) \end{aligned}$$

- Define $R : L^2(\partial\Omega) \to L^2(\partial\Omega)$ s.t. $Rf = g$

$$R = (T_{\text{int}} - i\eta)(T_{\text{int}} + i\eta)^{-1} \qquad (7)$$

- For real η, real $b(\boldsymbol{x})$ and self-adjoint T_{int}, R is unitary!

ItI maps

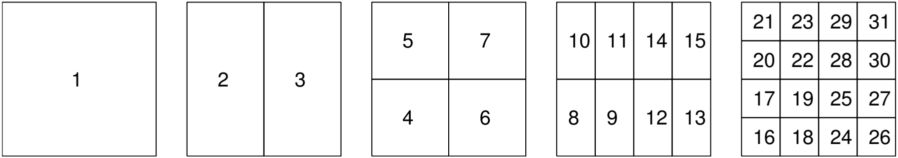


Figure: Ω split into boxes.

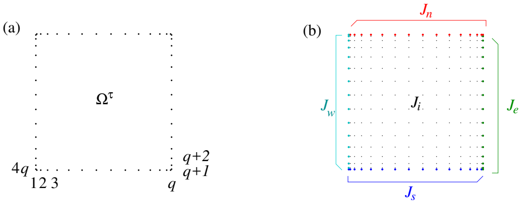


Figure: Operators on a leaf box

Leaf box operations

- Constructs $4q$ Gauss-Legendre edge grid and an internal $p \times p$ Chebyshev grid (careful indexing)
- Construct discretized PDE (4) operators:

$$
\begin{aligned}
\mathsf{A} &= (\mathsf{D}^{(1)})^2 + (\mathsf{D}^{(2)})^2 + diag\{\kappa^2(1 - b(\boldsymbol{x}_j))\}_{j=1}^{p^2} && (8)\\
\mathsf{F} &= \mathsf{N} + i\eta \mathsf{I}_p^2(J_b, :) \rightarrow \text{Impedance operator} && (9)\\
\mathsf{B} &= \begin{bmatrix} \mathsf{F} \\ \mathsf{A}(J_i, :) \end{bmatrix} && (10)
\end{aligned}
$$

- Construct a "solution matix", X (basis) for the B operator:

$$\mathsf{BX} = \begin{bmatrix} \mathsf{I}_{4p-4} \\ 0_{(p-2)^2 \times (4p-4)} \end{bmatrix}$$

- Interpolate X from Chebyshev to Gauss points using P, $\mathsf{Y} = \mathsf{XP}$.
- Define G similar to F (but on Gauss points), which gives:

$$\mathsf{R} = \mathsf{QGY}, \quad \mathsf{Q} \rightarrow \text{Gauss to Chebyshev} \qquad (11)$$

Merging leaf boxes

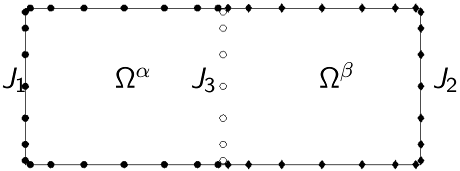


Figure: Merging operators for children α and β

- If $f^{\alpha,\beta}$ and $g^{\alpha,\beta}$ are the impedance traces:

$$\begin{bmatrix} R^\alpha_{11} + R^\alpha_{13} W R^\beta_{33} R^\alpha_{31} & -R^\alpha_{13} W R^\beta_{32} \\ -R^\beta_{23}\left(R^\alpha_{31} + R^\alpha_{33} W R^\alpha_{33} R^\alpha_{31}\right) & R^\beta_{22} + R^\beta_{23} R^\alpha_{33} Q R^\beta_{32} \end{bmatrix} \begin{bmatrix} f^\alpha_1 \\ f^\beta_2 \end{bmatrix} = \begin{bmatrix} g^\alpha_1 \\ g^\beta_2 \end{bmatrix} \tag{12}$$

- Here, $W := \left(I - R^\beta_{33} R^\alpha_{33}\right)^{-1}$

$$T_{\text{int}} = -i\eta \left(R^1 - I\right)^{-1} \left(R^1 + I\right) \tag{13}$$

Exterior constant coefficient problem

- Any solution may be written as (Green's formula)[4]:

$$u^{s}(\boldsymbol{x}) = \left(\mathcal{D}u^{s}|_{\partial\Omega}\right)(\boldsymbol{x}) - \left(\mathcal{S}u^{s}_{n}\right)(\boldsymbol{x}), \qquad \text{for } \boldsymbol{x} \in \Omega^{c}, \tag{14}$$

  where $(\mathcal{D}\phi)(\boldsymbol{x}) := \int_{\partial\Omega} \frac{\partial}{\partial n_{\boldsymbol{y}}}\left(\frac{i}{4}H_0^{(1)}(\kappa|\boldsymbol{x}-\boldsymbol{y}|)\right)\phi(\boldsymbol{y})ds_{\boldsymbol{y}}$ and $(\mathcal{S}\phi)(\boldsymbol{x}) := \int_{\partial\Omega} \frac{i}{4}H_0^{(1)}(\kappa|\boldsymbol{x}-\boldsymbol{y}|)\phi(\boldsymbol{y})ds_{\boldsymbol{y}}$
- Final formulation:

$$\left(\tfrac{1}{2}I - D + ST_{\text{int}}\right)u^{s}|_{\partial\Omega} = S\left(u^{i}_{n} - T_{\text{int}}u^{i}|_{\partial\Omega}\right) \tag{15}$$

- Use Nyström method with composite (panel-based) quadrature with $n \approx \sqrt{N}$ nodes in total.

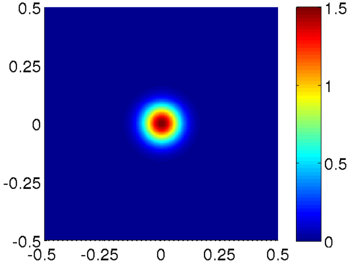
[4] Colton, Kress, Inverse acoustic and Electromagnetic Scattering Theory

Complexity

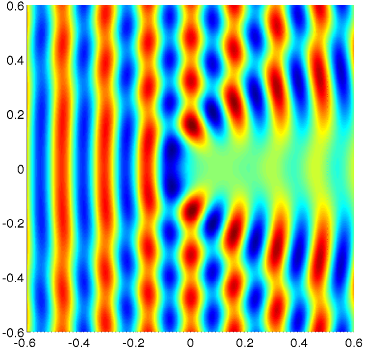
- Leaf solution matrix $\sim O(p^6) \times k$ boxes
- Compute $R \sim O(N^{3/2})$
- Applying $T_{\text{int}} \sim O(N)$
- Approximating $T_{\text{int}} \sim O(N^{3/2})$
- Quadrature $\sim O(N^{3/2})$ (GMRES convergence in $O(1)$ iterations)

Results (to come soon!)

- On a Gaussian bump scattering potential:



(a) Bump scattering potential, $b(\boldsymbol{x}) = 1.5e^{-160r^2}$



(b) Re(u)

Figure: Result for toy problem, with $N = 231361$, $n = 1760$, error $\approx 5e - 10$