A Fast Stokes Solver

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Stokes PDEs

$$\mu\nabla^2 u - \nabla p + f = 0$$
$$\nabla \cdot u = 0$$

where u is the velocity vector, μ is the viscosity, p is the pressure and f is the force.

Stokes PDEs: 3D case

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}+f_x=0$$
$$\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\partial p}{\partial y}+f_y=0$$
$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\partial p}{\partial z}+f_z=0$$
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Green's Function solution: The Stokeslet

$$u(r) = F \cdot \mathbb{J}(r)$$
$$p(r) = \frac{F \cdot r}{4\pi |r|^3}$$

where

$$r = \begin{bmatrix} x & y & z \end{bmatrix}$$
$$\mathbb{J}(r) = \frac{1}{8\pi\mu}\left(\frac{\mathbb{I}}{|r|} + \frac{rr}{|r|^3}\right)$$

Green's Function solution: The Stokeslet

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = \frac{1}{8\pi\mu|r|^3} \begin{bmatrix} |r|^2 + x^2 & xy & xz \\ xy & |r|^2 + y^2 & yz \\ xz & yz & |r|^2 + z^2 \\ 2\mu x & 2\mu y & 2\mu z \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Current implementation

pytential uses 9 kernel functions. (Reuse 3 kernels)

Drawbacks of this method are,

- Takes more storage than necessary
- Redundant computations
  - Derivative relationships in Taylor series
  - Common Subexpression Elimination

Revisiting local expansion

Let x be the targets and y be the sources and f be the charge/force

$$f(y)\phi(x-y) \approx \sum_{|p|\leq k} \frac{D_x^p f(y)\phi(x-y)\big|_{x=c}}{p!}(x-c)^p$$

Here $\frac{f(y)D_x^p\phi(x-y)\big|_{y=c}}{p!}$ depends on source and center, while $(x-c)^p$ depends on target and center

Local expansion for Stokes

In the stokes equation case u is a linear combination of 3 functions ϕ_1, ϕ_2, ϕ_3 and there are 3 charges f_1, f_2, f_3

$$u(x-y) = \sum_{i=1}^{3} f_i(y)\phi_i(x-y)$$

$$\approx \sum_{|p|\leq k} \frac{D_x^p \left(\sum_{i=1}^{3} f_i(y)\phi_i(x-y)\right)\Big|_{x=c}}{p!}(x-c)^p$$

Revisiting multipole expansion

Let x be the targets and y be the sources and f be the charge/force

$$f(y)\phi(x-y) \approx f(y) \sum_{|p|\le k} \frac{D_x^p \phi(x-y)\big|_{y=c}}{p!}(y-c)^p$$

or equivalently

$$f(y)\phi(x-y) \approx \sum_{|p|\le k} \frac{D_x^p \phi(x-y)\big|_{y=c}}{p!} f(y)(y-c)^p$$

Here $\frac{D_x^p \phi(x-y)\big|_{y=c}}{p!}$ depends on target and center, while $f(y)(y-c)^p$ depends on source and center

In the stokes equation case conside the first velocity component u which is a linear combination of 3 functions ϕ_1, ϕ_2, ϕ_3 and there are 3 charges f_1, f_2, f_3

$$u(x-y) = \sum_{i=1}^{3} f_i(y)\phi_i(x-y) \approx \sum_{i=1}^{3} \left(f_i(y) \sum_{|p|\leq k} \frac{D_x^p \phi_i(x-y)\big|_{y=c}}{p!}(y-c)^p \right)$$

$$u(x-y) \approx \sum_{|p|\leq k} \left(\sum_{i=1}^{3} \frac{D_x^p \phi_i(x-y)\big|_{y=c}}{p!} f_i(y)(y-c)^p \right)$$

Needs 12 multipoles

Multipole to local translation for Stokes

Let $\phi_{1,1}, \phi_{1,2}, \phi_{1,3}, \phi_{2,1}, \ldots, \phi_{3,3}$ be the 12 multipole kernels. Then,

$$u = \phi_{1,1} + \phi_{1,2} + \phi_{1,3}$$
$$v = \phi_{2,1} + \phi_{2,2} + \phi_{2,3}$$
$$w = \phi_{3,1} + \phi_{3,2} + \phi_{3,3}$$
$$p = \phi_{4,1} + \phi_{4,2} + \phi_{4,3}$$

Stokes PDE Derivatives

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial p}{\partial x} + f_x = 0 \quad (1)$$

$$\mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\partial p}{\partial y} + f_y = 0 \quad (2)$$

$$\mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\partial p}{\partial z} + f_z = 0 \quad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

Stokes PDE Derivatives

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}+f_x=0 \tag{1}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\partial p}{\partial y}+f_y=0 \tag{2}$$

$$\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)-\frac{\partial p}{\partial z}+f_z=0 \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4}$$

$\frac{\partial(1)}{\partial x}+\frac{\partial(2)}{\partial y}+\frac{\partial(3)}{\partial z}$ gives us

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=0 \tag{5}$$

Pressure term satisfies the Laplace equation !

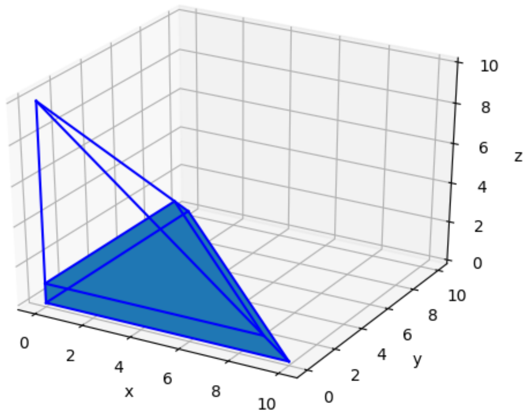
Laplace Derivative Reduction

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0$$

$$\frac{\partial^{a+b+2} p}{\partial x^{a+2} y^b} + \frac{\partial^{a+b+2} p}{\partial x^a y^{b+2}} + \frac{\partial^{a+b+2} p}{\partial x^a y^b z^2} = 0$$

$$\frac{\partial^{a+b+2} p}{\partial x^a y^b z^2} = -\frac{\partial^{a+b+2} p}{\partial x^{a+2} y^b} - \frac{\partial^{a+b+2} p}{\partial x^a y^{b+2}}$$

Laplace Derivative Reduction



Stokes Derivative Reduction

$$\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\partial p}{\partial x} + f_x = 0 \tag{1}$$

$$\frac{\partial^{a+b+2} u}{\partial x^a y^b z^2} = \frac{1}{\mu} \left(\frac{\partial^{a+b+1} p}{\partial x^{a+1} y^b} - \frac{\partial^{a+b+2} u}{\partial x^{a+2} y^b} - \frac{\partial^{a+b+2} u}{\partial x^a y^{b+2}} \right) \tag{a}$$

Stokes Derivative Reduction

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}+f_x=0 \tag{1}$$

$$\frac{\partial^{a+b+2}u}{\partial x^a y^b z^2}=\frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a+1}y^b}-\frac{\partial^{a+b+2}u}{\partial x^{a+2}y^b}-\frac{\partial^{a+b+2}u}{\partial x^a y^{b+2}}\right) \tag{a}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\partial p}{\partial y}+f_y=0 \tag{2}$$

$$\frac{\partial^{a+b+2}v}{\partial x^a y^b z^2}=\frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a}y^{b+1}}-\frac{\partial^{a+b+2}v}{\partial x^{a+2}y^b}-\frac{\partial^{a+b+2}v}{\partial x^a y^{b+2}}\right) \tag{b}$$

Stokes Derivative Reduction

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\partial p}{\partial x}+f_x=0 \tag{1}$$

$$\frac{\partial^{a+b+2}u}{\partial x^a y^b z^2}=\frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a+1}y^b}-\frac{\partial^{a+b+2}u}{\partial x^{a+2}y^b}-\frac{\partial^{a+b+2}u}{\partial x^a y^{b+2}}\right) \tag{a}$$

$$\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\partial p}{\partial y}+f_y=0 \tag{2}$$

$$\frac{\partial^{a+b+2}v}{\partial x^a y^b z^2}=\frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a}y^{b+1}}-\frac{\partial^{a+b+2}v}{\partial x^{a+2}y^b}-\frac{\partial^{a+b+2}v}{\partial x^a y^{b+2}}\right) \tag{b}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{4}$$

$$\frac{\partial^{a+b+1}w}{\partial x^a y^b z}=-\frac{\partial^{a+b+1}u}{\partial x^{a+1}y^b}-\frac{\partial^{a+b+1}v}{\partial x^a y^{b+1}} \tag{c}$$

Stokes Derivative Reduction

$$\frac{\partial^{a+b+2}u}{\partial x^a y^b z^2} = \frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a+1}y^b} - \frac{\partial^{a+b+2}u}{\partial x^{a+2}y^b} - \frac{\partial^{a+b+2}u}{\partial x^a y^{b+2}}\right) \quad \text{(a)}$$

$$\frac{\partial^{a+b+2}v}{\partial x^a y^b z^2} = \frac{1}{\mu}\left(\frac{\partial^{a+b+1}p}{\partial x^{a}y^{b+1}} - \frac{\partial^{a+b+2}v}{\partial x^{a+2}y^b} - \frac{\partial^{a+b+2}v}{\partial x^a y^{b+2}}\right) \quad \text{(b)}$$

$$\frac{\partial^{a+b+1}w}{\partial x^a y^b z} = -\frac{\partial^{a+b+1}u}{\partial x^{a+1}y^b} - \frac{\partial^{a+b+1}v}{\partial x^a y^{b+1}} \quad \text{(c)}$$

$$\frac{\partial^{a+b+2}p}{\partial x^a y^b z^2} = -\frac{\partial^{a+b+2}p}{\partial x^{a+2}y^b} - \frac{\partial^{a+b+2}p}{\partial x^a y^{b+2}} \quad \text{(d)}$$

Stokeslet Kernel Derivative Reduction

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = \frac{1}{8\pi\mu|r|^3} \begin{bmatrix} |r|^2 + x^2 & xy & xz \\ xy & |r|^2 + y^2 & yz \\ xz & yz & |r|^2 + z^2 \\ 2\mu x & 2\mu y & 2\mu z \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$\begin{bmatrix} u & v & w & p \end{bmatrix}$ satisfies the Stokes PDEs

Stokeslet Kernel Derivative Reduction

$$\begin{bmatrix} u \\ v \\ w \\ p \end{bmatrix} = \frac{1}{8\pi\mu|r|^3} \begin{bmatrix} |r|^2 + x^2 & xy & xz \\ xy & |r|^2 + y^2 & yz \\ xz & yz & |r|^2 + z^2 \\ 2\mu x & 2\mu y & 2\mu z \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$\begin{bmatrix} u & v & w & p \end{bmatrix}$ satisfies the Stokes PDEs

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ p_0 \end{bmatrix} = \frac{1}{8\pi\mu|r|^3} \begin{bmatrix} |r|^2 + x^2 \\ xy \\ xz \\ 2\mu x \end{bmatrix}$$

$\begin{bmatrix} u_0 & v_0 & w_0 & p_0 \end{bmatrix}$ satisfies the Stokes PDEs

Questions?