Solving Elliptic (and Hyperbolic) Differential Equations in Nonlinear Viscoelasticity

Elasticity $\longrightarrow$ Hyperelasticity $\longrightarrow$ Viscoelasticity

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Introduction: Continuum Mechanics

Kinematics

Deformation mapping (χ) and Deformation Gradient ($\mathbf{F}$)

$$\exists \chi(\mathbf{X}) \in C^2(\Omega_0) : \begin{cases} \mathbf{F} & = \nabla\chi \equiv F_{ij} = \dfrac{\partial \chi_i}{\partial X_j} = \dfrac{\partial x_i}{\partial X_j}, \qquad 1 \leq i,j \leq 3 \\ J & = \det\mathbf{F} > 0 \quad \text{also} \quad \mathbf{u} = \chi - X \implies \boxed{\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}} \end{cases}$$

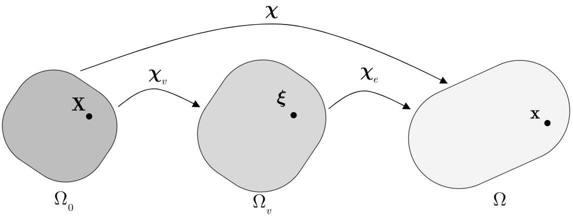
It is difficult to analytically determine χ for most BVPs (**Semi-inverse method, Fourier)** or (**FEM,BEM**!)

Newton's 2$^{\text{nd}}$ Law

Stresses (Cauchy and Piola-Kirchoff)

$$\exists \mathbf{T} : \mathbf{t} = \mathbf{Tn} \quad \& \quad \boxed{\int_\Omega \mathbf{b}(\mathbf{x},t)\,d\mathbf{x} + \int_{\partial\Omega} \mathbf{t}(\mathbf{x},t)\,d\mathbf{x} = \int_\Omega \rho(\mathbf{x},t)\,\ddot{\chi}(\mathbf{x},t)\,d\mathbf{x}}$$

More Continuum Mechanics...



$$\mathbf{A}^T = \mathbf{A}$$
$$\mathbf{B}^T = \mathbf{B}$$
$$\therefore \mathbf{AB} = \mathbf{BA}$$

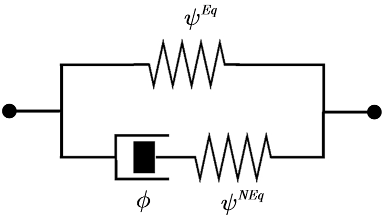
$$\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$$

- Modeling Viscoelasticity – Two approaches
  - Hereditary Integrals: Stieltjes Integral
  - Internal variables (**Increasingly popular!**)
- **Two Potential Constitutive Framework:** ψ and ϕ

$$\text{Constitutive Model:} \begin{cases} \mathbf{S}\left(\mathbf{F},\mathbf{F}^v\right) = \dfrac{\partial\psi}{\partial\mathbf{F}}\left(\mathbf{F},\mathbf{F}^v\right) \\ \dfrac{\partial\psi}{\partial\mathbf{F}^v} + \dfrac{\partial\phi}{\partial\dot{\mathbf{F}}^v} = \mathbf{0} \end{cases} \quad \& \quad \underbrace{\text{Div}\mathbf{S} + \mathbf{B} = \mathbf{0}}_{\text{BLM}} \qquad (1)$$

BVP

- Isotropy and Non-negativity



$$\psi(\mathbf{F}, \mathbf{F}^v) > 0$$
$$\psi(\mathbf{F}, \mathbf{F}^v) = \psi(\mathbf{QFK}, \mathbf{F}^v) \ \forall, \ \mathbf{Q}, \mathbf{K} \in \mathcal{U}$$
$$\mathcal{U} = \left\{\mathbf{A} : \mathbf{AA}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}\right\}$$

Given a free energy function (ψ) and dissipation potential (ϕ), a domain (Ω_0) with smooth boundary ($\partial\Omega_0$), choose an internal variable ($\mathbf{F}^v$) and solve :

$$\text{Div}\mathbf{S} = \mathbf{0} \quad \text{for} \quad \mathbf{X} \in \Omega_0 \tag{2}$$
$$\frac{\partial\psi}{\partial\mathbf{F}^v} + \frac{\partial\phi}{\partial\dot{\mathbf{F}}^v} = \mathbf{0} \quad \text{at each time step} \tag{3}$$

In general, the practice is to solve (3) at each time step (**discretization**) and then solve (2) using **FEM**

Hyperelasticity ($\phi = 0$)

For now, consider no dissipation and the following (ψ) (**Convex**!)

$$\psi = \frac{\mu}{2}(I_1 - 3) + \frac{\kappa}{2}(J-1)^2 \quad \text{where} \quad I_1 = \mathbf{F}\cdot\mathbf{F} \equiv F_{ij}F_{ij} \quad (\text{Neo-Hookean})$$

$$\implies \mathbf{S} = \mu\mathbf{F} + \kappa(J-1)J\mathbf{F}^{-T} \longleftarrow \begin{cases} \dfrac{\partial I_1}{\partial \mathbf{F}} & = \dfrac{\partial}{\partial \mathbf{F}}(\mathbf{F}\cdot\mathbf{F}) = 2\mathbf{F} \\ \dfrac{\partial J}{\partial \mathbf{F}} & = \dfrac{\partial}{\partial \mathbf{F}}(\det\mathbf{F}) = J\mathbf{F}^{-T} \end{cases}$$

Underlying PDE

By balance of linear momentum, we finally get the PDE

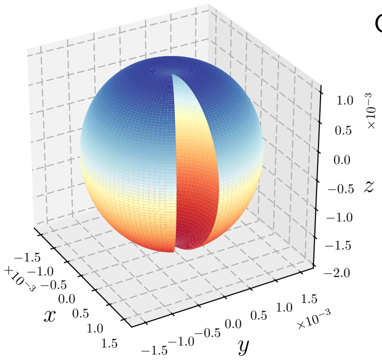
$$\text{Div}\mathbf{S} = \mathbf{0} \implies \mu\nabla\cdot\mathbf{F} + \kappa J(J-1)\nabla\cdot\mathbf{F}^{-T} = \mathbf{0}$$

$$\implies \mu\nabla^2\mathbf{u} + \kappa\nabla(J(J-1))\mathbf{F}^{-T} = \mathbf{0} \quad \text{with} \begin{cases} \mathbf{u} & = \mathbf{g} \quad \text{on } \partial\Omega_0^x \\ \mathbf{t} & = \mathbf{h} \quad \text{on } \partial\Omega_0^t \end{cases} \tag{4}$$

Equation (4) is the Cauchy-Navier equation for **Hyperelasticity**

BVP: Set up

- Quasi-static deformation of a spherical shell ($R = |\mathbf{X}|$)
- For now, consider ($J > 0$), later we will consider ($J = 1$)



Consider the domain on the left, given by

$$\Omega : \mathbf{X} \in \mathbb{R}^3, \; A \leq |\mathbf{X}| \leq B \tag{5}$$

$$\text{where} \quad \begin{cases} A &= 10^{-3} \\ B &= 2 \times 10^{-3} \end{cases} \tag{6}$$

Assumption: **Radially symmetric deformation**

- Points move radially outward
- Both **Dirichilet** and **Neumann**
- No bifurcations (Cavitation!)

Radially symmetric mappings

Consider deformation mapping (χ) of the form

$$\chi = f(R)\mathbf{X} \implies \mathbf{F} = (Rf'(R) + f)\underbrace{\frac{1}{R^2}\mathbf{X}\otimes\mathbf{X}}_{\mathcal{K}_1} + f\underbrace{\left(I - \frac{1}{R^2}\mathbf{X}\otimes\mathbf{X}\right)}_{\mathcal{K}_2} \tag{7}$$

$$\implies \mathbf{F} = \lambda_1\mathcal{K}_1 + \lambda_2\mathcal{K}_2 \iff \mathbf{S} = \sigma_1\mathcal{K}_1 + \sigma_2\mathcal{K}_2 \tag{8}$$

Matrix Forms

The spectral forms of **S** and **F**

$$\mathbf{S} = \begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_2\end{bmatrix} \quad \mathbf{F} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_2\end{bmatrix} \tag{9}$$

With some algebra, the BLM reduces to

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}R} + \frac{2}{R}(\sigma_1 - \sigma_2) = 0 \qquad \text{with} \quad f(A) = 1\ ,\ f(B) = 2 \tag{10}$$

BVP... Finally

Therefore, the entire problem reduces to a single nonlinear ODE of the form

$$4\left(\mu+\kappa f^4\right)+2R\kappa f^3 f'^2+R\left(\mu+\kappa f^4\right)f''=0 \tag{11}$$

which reduces to

$$f(R)+2\kappa\int_A^R K(R,f'(\xi),\xi)\,\mathcal{F}\left(f(\xi)\right)\,d\xi=\mathcal{G}(R) \tag{12}$$

$$\mathcal{G}(R)=1+c(R-A)-4A\left(\frac{R}{A}\left(\log\left(\frac{R}{A}\right)-1\right)+1\right) \tag{13}$$

$$K(R,f'(\xi),\xi)=(R-\xi)f'^2(\xi) \tag{14}$$

$$\mathcal{F}\left(f(\xi)\right)=\frac{f^3(\xi)}{\mu+\kappa f^4(\xi)} \tag{15}$$

$$c=1+4A\left(\frac{B}{A}\left(\log\left(\frac{B}{A}\right)-1\right)+1\right)+2\kappa\int_A^B K(B,f'(\xi),\xi)\mathcal{F}\left(f(\xi)\right)\,d\xi \tag{16}$$

Quadrature $\longrightarrow$ Nonlinear System

Using ideas from Linear IEs

- Nyström discretization
- n-point Gauss-Legendre Quadrature to evaluate the integrals in (12)
- **Nonlinear-Kernel**

$$f_n(R_i) = \mathcal{G}_n(R_i) - 2\kappa \sum_{j=1}^{N} \omega_j K\left(R_i, f_n'(\xi_j), \xi_j\right) \mathcal{F}\left(f_n(\xi_j)\right) \qquad (17)$$

Successive approximations

$$f_n^{(k+1)}(R_i) = \mathcal{G}_n^{(k)}(R_i) - 2\kappa \sum_{j=1}^{N} \omega_j K\left(R_i, f_n'^{(k)}(\xi_j), \xi_j\right) \mathcal{F}\left(f_n^{(k)}(\xi_j)\right) \qquad (18)$$

Existence (and Uniqueness)

Nature of $f'(R)$

- Exact form of the kernel not reported in the literature
- For equations of the following form

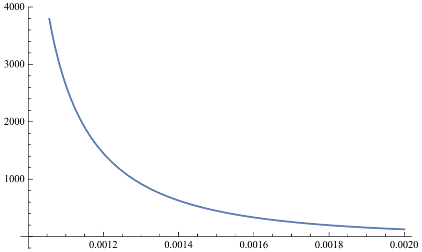
$$f(R) + \int_A^R K(R,\xi)\ \hat{\mathcal{F}}(f(\xi))\ d\xi = \mathcal{G}(R)$$

- $K(R,\xi)$ satisfies the Lipchitz condition
- $\hat{\mathcal{F}}$ satisfies Lipchitz condition
- $f(R)$ bounded and integrable
- $\mathcal{G}(R)$ bounded and integrable

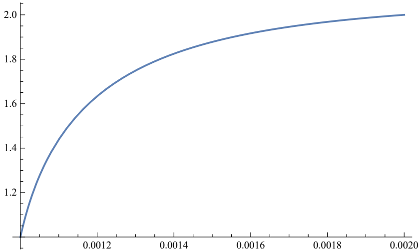
In general for nonlinear equations existence and uniqueness is not straightforward.

- Linearize (?)
- Do it for the linear problem

Sample Results



(a) $f'(R)$ vs R



(b) $f(R)$ vs R

Calculations from FEM

- The gradient is sharp as $R \longrightarrow A^+$
- Need more points to evaluate the integral (?)

THANK YOU